BRILL

Ptolemy's Pythagoreans, Archytas, and Plato's Conception of Mathematics
Author(s): Andrew Barker
Source: *Phronesis*, Vol. 39, No. 2 (1994), pp. 113-135
Published by: Brill
Stable URL: https://www.jstor.org/stable/4182463
Accessed: 12-05-2019 16:06 UTC

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at https://about.jstor.org/terms

Brill is collaborating with JSTOR to digitize, preserve and extend access to *Phronesis*

JSTOR

This content downloaded from 130.43.141.16 on Sun, 12 May 2019 16:06:32 UTC
All use subject to https://about.jstor.org/terms

Ptolemy's Pythagoreans, Archytas, and Plato's conception of mathematics

ANDREW BARKER

Introduction

Ptolemy's *Harmonics*[1] was probably written during the third quarter of the second century AD, but its repeated references to the work of Ptolemy's predecessors preserve much material that is considerably older. Some of it unquestionably goes back to the fourth century BC, hard though it often is to disentangle this earlier level of thought from more recent accretions.[2] The main tasks of this paper are to review a group of three arguments attributed by Ptolemy in *Harm.* I.5 to people he calls 'Pythagoreans', to argue that two of them very probably reflect the contents of fourth-century documents, and to name their author; and finally, to indicate some reasons why these conclusions, if they are acceptable, should be of interest to 'ancient philosophers' in general, not merely to specialists in mathematics or harmonic theory. More specifically, they may cast some light on the troublesome issue of the relations between mathematical expertise and an understanding of the nature of the good, as these are envisaged in Plato's *Republic*. I hope that the prospect of this dénoument will be enough to tempt faint philosophical hearts into tackling my paper's earlier and perhaps more arid stretches.

A little preliminary discussion is needed to set the arguments Ptolemy reports in their context in the tradition of scientific harmonics. Fortunately

[1] For the text see I. Düring, *Die Harmonielehre des Klaudios Ptolemaios* (Göteborg 1930, repr. N.Y. and London 1980): line and page numbers given below refer to that edition. There is a German translation in Düring's *Ptolemaios und Porphyrios über die Musik* (Göteborg 1934, repr. 1980), and an English version in A. Barker, *Greek Musical Writings* vol. 2 (Cambridge 1990) pp. 270-391 (cited below as *GMW* 2). I am grateful to Cambridge University Press for permission to quote at some length from this work. Porphyry's commentary on Ptolemy's *Harmonics* (hereafter 'Porph., *Comm.*') is printed in Düring's *Porphyrios Kommentar zur Harmonielehre des Ptolemaios* (Göteborg 1932, repr. 1980). Texts of several of the other works mentioned below are in *Musici Scriptores Graeci* ed. C. von Jan (Leipzig 1895), hereafter *MSG*, most of which is reproduced, with Italian translation and notes, in L. Zanoncelli, *La Manualistica musicale Greca* (Milan 1990).

[2] The views of Aristoxenus, for instance, are reported quite accurately for the most part, though very incompletely and with thoroughly hostile intent.

Phronesis 1994. Vol. XXXIX/2 (Accepted September 1993)

This content downloaded from 130.43.141.16 on Sun, 12 May 2019 16:06:32 UTC
All use subject to https://about.jstor.org/terms

we shall have to appeal to no very ferocious technicalities. We may begin from the famous, more or less empirical discoveries that had entered Greek musical discussions by the fifth century BC, and were usually attributed by later writers to Pythagoreans.[3] If you take a stretched string of even thickness and constitution, and pluck it, producing a pitched sound, and then divide it in half with a bridge and pluck one of the resulting sections, the second note sounded will be exactly an octave above the first. If instead you divide it two thirds of the way along, and pluck the longer section, the note sounded will be at the interval of a perfect fifth above the first pitch; and if you divide it three quarters of the way along and again pluck the longer section, the note will be at the interval of a perfect fourth above that sounded by the whole length. Hence the octave is associated with lengths in the ratio 2:1, the fifth with lengths in the ratio 3:2, and the fourth with lengths in the ratio 4:3.

To Greek musicologists these facts were very significant. The intervals of the octave, fifth and fourth were fundamental to all Greek forms of attunement, providing them with a structural foundation in a way that we need not now pursue.[4] But secondly, these intervals were held to strike the ear in a special way, characteristic of no other interval within that range: only they were given the title 'concords', συμφωνίαι. The attribute of concordance, as the ear perceives it, was regularly analysed (at least from the fourth century) as a particularly intimate kind of 'blending' between two sounds. In these relations, it was said, and in no others, two sounds are heard not as two different things lying side by side, as other pairs of notes are, but as a single thing, its theoretically distinct elements fused into a unity.[5] It is not surprising, then, that speculations about the ways in which diverse items of any sort could be organised to form a coherent whole, one thing from many, commonly took the peculiar acoustic and mathematical properties of these musical intervals as their inspiration and starting point.[6]

[3] See for instance Theon Smyrn. 59.4-21 = DK 18.13, schol. to Plato *Phaedo* 108d4 = DK 18.12, Nicomachus *Enchiridion* 6 (in *MSG*; closely paraphrased by Iamblichus *Vit. Pyth.* 26): translations in *GMW* 2.

[4] See most recently and straightforwardly M.L. West, *Ancient Greek Music* (Oxford 1992) pp. 160-1 (together, if necessary, with his aids to musical beginners on pp. 8-10).

[5] E.g. Plato *Tim.* 80b, Arist. *De an.* 426b, *De sensu* 447a-b, 448a, Eucl. *Sect. Can.* 149.17 (*MSG*). These ideas form the basis of the formal definitions of concordance offered by later musical theorists, e.g. Nicomachus *Ench.* 262.1, Cleonides 187.19, Bacchius 293.8, Gaudentius 337.8 (all in *MSG*), cf. Aristides Quintilianus *De Mus.* 10.1 (Winnington-Ingram).

[6] See e.g. Philolaus as cited by Stobaeus *Anth.* i. 21, 7d = DK44 B6, Plato *Symp.* 187a-e, *Crat.* 405c-d, *Rep.* 432a (and many other passages in the dialogues), Arist. *De Sensu* 439-440a, 448a.

This content downloaded from 130.43.141.16 on Sun, 12 May 2019 16:06:32 UTC
All use subject to https://about.jstor.org/terms

By the early fourth century there had been two major developments. First, ideas in theoretical physics had been harnessed to support the view that pitch itself is a quantitative variable, and that the relations between pitches correspond inversely to the relevant dimensions of string, pipe or whatever it may be that sounds them.[7] For present purposes we can again by-pass the task of analysing the arguments and the force of their conclusions in detail. We need only note the crucial consequence that pitches themselves, and not merely the dimensions of sound-producing objects, can be described as standing to one another in numerical ratios, these ratios being the mirror-images of those between the relevant lengths of string.[8] Thus the higher note in the octave stands to the lower in the ratio 2:1, and so on.

Secondly, the focus began to shift from empirical observations to the more difficult task of accounting for them. In particular, there were attempts by Archytas and Plato, and perhaps by some earlier writers too,[9] to explain what it is about these numerical ratios, simply as such, that marks them out as mathematically special. Since it is because they have these ratios that certain intervals are concordant, the perceptible coherence and unity of each concord must be a reflection of the privileged sort of mathematical coherence or 'concordance' that characterises the corresponding ratio. What then is this special kind of mathematical coherence? To what mathematical principles do these ratios conform while others do not? It is because of the importance of these questions that we find Plato's Socrates insisting in *Republic* VII (531c) that students of harmonics should investigate 'which *numbers*' (rather than just which sounds) 'are concordant and which are not, and in each case why'.

Whatever their date, it is to this sort of enquiry that the arguments we shall be considering belong. Their direct purpose goes a step further. Once the basic principles governing mathematical 'concordance' have been established, they are immediately put to work, to show why it is that the three primary musical concords must have exactly the ratios they do have, 2:1,

[7] Especially Archytas ap. Porph. *Comm. 56.5-57.27* = DK47 B1, Plato *Tim.* 79e-80b, Theophrastus ap. Porph. *Comm.* 61.16-65.15 (fr. 89 Wimmer = 716 Fortenbaugh et al.), all tr. in *GMW* 2. For a discussion of the limiting case in which relations between notes or pitches just *are* relations between numbers, see A.C. Bowen, 'Euclid's *Sectio Canonis* and the history of Pythagoreanism' in *Science and Philosophy in Classical Greece* ed. Bowen (N.Y. and London 1991), especially pp. 172-82.

[8] The point is made by e.g. Thrasyllus ap. Theon Smyrn. 87.10ff, Nicomachus *Ench.* 254.5-13 (*MSG*), Ptol. *Harm.* 8.21 (all in *GMW* 2).

[9] Archytas and Plato will be discussed below. For the earlier investigations see Porph. *Comm.* 107.15ff = DK47 A17 (tr. in *GMW* 2).

This content downloaded from 130.43.141.16 on Sun, 12 May 2019 16:06:32 UTC
All use subject to https://about.jstor.org/terms

3:2, 4:3; and also which other intervals, larger than these, can be counted by these principles as concordant, and what their ratios are. There are two main arguments, which I shall call (a) and (b), and a short appendix attached by Ptolemy to (b), which I shall treat separately and call (c).

SECTION 1 The three 'Pythagorean' arguments of *Harm.* I.5

(i) Argument (a): *Harm.* 11.8-12.7

[A] They laid down a first principle of their method that was entirely appropriate, according to which equal numbers should be associated with equal-toned notes, and unequal numbers with unequal-toned; [B] and from this they argue that just as there are two primary classes of unequal-toned notes, that of the concords and that of the discords, and that of the concords is finer, so there are also two primary distinct classes of ratio between unequal numbers, one being that of what are called 'epimeric' or 'number to number' ratios, the other being that of the epimorics and multiples; and of these the latter is better than the former [C] on account of the simplicity of the comparison, since in it the excess, in the case of epimorics, is a simple part, while in the multiples the smaller term is a simple part of the greater. [D] For this reason they fit the epimorics and multiple ratios to the concords, and link the octave to duple ratio (2:1), the fifth to hemiolic (3:2), the fourth to epitritic (4:3). [E] Their procedure here is very rational, since the octave is the finest of the concords, and the duple is the best of the ratios, the former because it is nearest to the equal-toned, the latter because it alone makes the excess equal to that which is exceeded; [F] and again, because the octave consists of the first two concords taken successively, and the duple consists of the two first epimorics taken successively, the hemiolic and the epitritic; and while in the latter case the hemiolic ratio is greater than the epitritic, in the former the concord of the fifth is greater than that of the fourth, [G] so that the difference between them – that is, the tone – is assigned to the epogdoic ratio (9:8), by which the hemiolic is greater than the epitritic; and in accordance with these points they also adopt among the concords the magnitude put together from the octave and the fifth, and again that put together from two octaves – that is, the double octave – since it follows that the ratio of the latter is constituted as quadruple, and that of the former as triple. [H] But they do not adopt the magnitude put together from the octave and the fourth, because it makes the ratio of 8 to 3, which is neither epimoric nor multiple. [10]

This content downloaded from 130.43.141.16 on Sun, 12 May 2019 16:06:32 UTC
All use subject to https://about.jstor.org/terms

What Ptolemy calls the 'first principle' (ἀρχή) of this argument, at the beginning of [A], is the simple rule that in quantitative representations of musical relations, notes equal in pitch should be assigned equal numbers, and unequal pitches unequal numbers. The next step, [B], is to argue that just as there are two primary forms or classes, εἴδη, of relation between notes of unequal pitch, so there are two primary and distinct types, διαφοραί, of ratios between numbers. The two forms of pitch-relation are concord and discord, and concord is the 'finer' or 'more beautiful', κάλλιον: the two classes of ratio, it says, are multiples and epimorics on the one hand and epimerics or 'number to number' ratios on the other; [11] and of

[10] The text is as follows.

ἀρχὴν γὰρ οἰκειοτάτην ποιησάμενοι τῆς μεθόδου, καθ' ἣν οἱ μὲν ἴσοι τῶν ἀριθμῶν παραβληθήσονται τοῖς ἰσοτόνοις φθόγγοις, οἱ δὲ ἄνισοι τοῖς ἀνισοτόνοις, τοὐντεῦθεν ἐπάγουσιν, ὅτι καθάπερ τῶν ἀνισοτόνων φθόγγων δύο ἐστὶν εἴδη πρὸς ἄλληλα τὰ πρῶτα, τό τε τῶν συμφώνων καὶ τῶν διαφώνων, καὶ κάλλιον τὸ τῶν συμφώνων, οὕτως καὶ τῶν ἀνίσων ἀριθμῶν δύο γίνονται πρῶται διαφοραὶ λόγων, μία μὲν ἡ τῶν λεγομένων ἐπιμερῶν καὶ ὡς ἀριθμὸς πρὸς ἀριθμόν, ἑτέρα δὲ ἡ τῶν ἐπιμορίων τε καὶ πολλαπλασίων, ἀμείνων καὶ αὕτη τῆς ἐκείνων κατὰ τὴν ἁπλότητα τῆς παραβολῆς, ὅτι μέρος ἐστὶν ἁπλοῦν ἐν αὐτῇ τῶν μὲν ἐπιμορίων ἡ ὑπεροχή, τῶν δὲ πολλαπλασίων τὸ ἔλαττον τοῦ μείζονος. ἐφαρμόσαντες δὴ διὰ τοῦτο τοὺς ἐπιμορίους καὶ πολλαπλασίους λόγους ταῖς συμφωνίαις, τὴν μὲν διὰ πασῶν προσάπτουσι τῷ διπλασίῳ λόγῳ, τὴν δὲ διὰ πέντε τῷ ἡμιολίῳ, τὴν δὲ διὰ τεσσάρων τῷ ἐπιτρίτῳ. λογικώτερον μὲν ἐπιχειροῦντες, ἐπειδὴ τῶν τε συμφωνιῶν ἡ διὰ πασῶν ἐστι καλλίστη καὶ τῶν λόγων ὁ διπλάσιος ἄριστος, ἡ μὲν διὰ τὸ ἐγγυτάτω εἶναι τοῦ ἰσοτόνου, ὁ δὲ τῷ μόνος τὴν ὑπεροχὴν ἴσην ποιεῖν τῷ ὑπερεχομένῳ, καὶ ἔτι τὴν μὲν διὰ πασῶν συγκεῖσθαι συμβέβηκεν ἐκ δύο τῶν ἐφεξῆς καὶ πρώτων συμφωνιῶν, τῆς τε διὰ πέντε καὶ τῆς διὰ τεσσάρων, τὸν δὲ διπλάσιον ἐκ δύο τῶν ἐφεξῆς καὶ πρώτων ἐπιμορίων, τοῦ τε ἡμιολίου καὶ τοῦ ἐπιτρίτου, μείζονα δὲ ἐνταῦθα μὲν τοῦ ἐπιτρίτου τὸν ἡμιόλιον λόγον, ἐκεῖ δὲ τῆς διὰ τεσσάρων τὴν διὰ πέντε συμφωνίαν, ὥστε καὶ τὴν ὑπεροχὴν αὐτῶν, τουτέστι τὸν τόνον, τίθεσθαι κατὰ τὸν ἐπόγδοον λόγον, ᾧ μείζων ἐστὶν ὁ ἡμιόλιος τοῦ ἐπιτρίτου, ἀκολούθως δὲ τούτοις καὶ τὸ μὲν ἐκ τῆς διὰ πασῶν καὶ τῆς διὰ πέντε συντιθέμενον μέγεθος καὶ ἔτι τὸ ἐκ δύο τῶν διὰ πασῶν, τουτέστι τὸ δὶς διὰ πασῶν, παραλαμβάνοντες εἰς τὰς συμφωνίας, ὅτι ταύτης μὲν ἀκολουθεῖ τὸν λόγον συνίστασθαι τετραπλάσιον, ἐκείνης δὲ τριπλάσιον, τὸ δ' ἐκ τῆς διὰ πασῶν καὶ τῆς διὰ τεσσάρων οὐκέτι διὰ τὸ ποιεῖν λόγον τὸν τῶν ὀκτὼ πρὸς τὰ τρία, μήτε ἐπιμόριον ὄντα μήτε πολλαπλάσιον.

[11] Terms are said to be in multiple ratio if the smaller term is an integral factor of the greater; in epimoric (or 'superparticular') ratio if the greater term is the sum of the smaller term and an integral factor of the smaller term. (That is the technical definition. Epimoric ratios are more easily recognised, however, by the fact that they are all of the form (n+1):n. The only ratio of that form which is not properly speaking epimoric is the

This content downloaded from 130.43.141.16 on Sun, 12 May 2019 16:06:32 UTC
All use subject to https://about.jstor.org/terms

these the former kind, that of the multiples and epimorics, is 'better', ἀμείνων. These Pythagorean theorists therefore associate concords, the 'finer' intervals, exclusively with multiples and epimorics, the 'better' ratios.

One might well pause here to wonder, in the first place, by what form of reasoning these contentions about forms of interval and types of ratio are supposed to be linked with the argument's starting point, its ἀρχή, which merely associated equality and inequality of pitch with equality and inequality of number. The relation is obviously not deductive. In his transition to [B] Ptolemy connects the ἀρχή to its sequel with the phrase τοὐντεῦθεν ἐπάγουσιν (neutrally translated above as 'from this they argue'), which hints at the procedure of ἐπαγώγη, 'induction' or 'abstraction'. Probably the movement of thought is something like this. If pitch relations are in essence relations of a quantitative sort, ultimately ratios of numbers, then we are entitled to expect that significant perceptible distinctions between classes of pitch-relation will be mirrored in equally significant mathematical distinctions between classes of ratio. More specifically here, where certain classes of pitch-relation are agreed to be aesthetically 'finer' than others, we should expect that their ratios will also be of higher status, in some way, from an intrinsically mathematical point of view. And finally, it is to be expected that the mathematical properties of the privileged ratios are such that they not only correspond to the aesthetic qualities of the perceptible pitch-relations, but also explain them. These are bold hypotheses. They are plausible, though not proven, in the light of the ἀρχή; and they underpin Ptolemy's own approach to harmonics as much as that of his putatively Pythagorean predecessors.

Multiple and epimoric ratios, we have been told, are 'better' than epimerics; and in [C] we are immediately offered an explanation of why this is so. It is 'because of the simplicity of the comparison' between the terms of the 'better' class of ratio. The idea is then briefly unpacked. The comparison is simple, 'because in it the excess (i.e. the difference between the terms) in the case of epimorics is a simple part (i.e. an integral factor of each of the terms), and in multiples the smaller term is a simple part of the greater'. Suppose we are comparing the sizes of two straightforwardly observable objects, the lengths, for example, of two straight sticks. To do so, we need either to express one length as a function of the other, or to express both as functions of some third length that we can identify and use as a 'measure'.

ratio 2:1, since the difference between the terms is in this case equal to the lowest term, and not a factor of it. Hence 2:1 is consistently treated as multiple rather than epimoric.) Every ratio that is neither multiple nor epimoric is for present purposes classified as epimeric.

This content downloaded from 130.43.141.16 on Sun, 12 May 2019 16:06:32 UTC
All use subject to https://about.jstor.org/terms

Now if the ratio between their lengths is multiple, we can use the former strategy: the longer stick is so many times the length of the shorter. If it is epimoric, we can use the latter: we use as our 'measure' the difference between their lengths, which is readily identified, and each stick's length is some integral number of times that measure. If the ratio is epimeric, on the other hand, there is no length immediately given to the observer's perception that can be used as a measure, a length such that each of the sticks is so many times its size. Hence in the case of epimerics the business of comparing the quantities is bound to be more complex and difficult.[12]

This characteristic of multiples and epimorics can fairly be construed as a kind of 'simplicity'; and one can have some sympathy with the idea that the simplicity of a relation is in some sense a mathematical excellence, though the grounds of this intuition are not altogether clear. It is much less obvious that it is the same sort of excellence as is presented to perception in the concords, and that we are justified in claiming that the former is the formal counterpart of the latter and can serve to explain it. From Ptolemy's discussions elsewhere in the *Harmonics* it is possible to extract, rather painfully, a group of notions apparently deployed to make intelligible the correlation of 'simplicity of comparison' in ratios with a corresponding simplicity of comparison between pitches that are heard as concordant.[13] But these notions are thoroughly idiosyncratic, and there is no trace of them in the present, 'Pythagorean' argument. Nor is there any suggestion here that the honorific attributes of being 'finer' and 'better' can be eliminated, in favour of some more positivistic analysis in which the relevant attributes of the ratios and of the concords are shown to be the same. The correlation of concords with multiple and epimoric ratios depends wholly on the assumption that what is aesthetically finer goes with what is mathematically better: without these evaluative characterisations no connection would exist, and the argument could not proceed. This point needs emphasis. It will turn out to be of substantial importance later in the paper.

The next phase of the argument can be summarised quite briefly. We have reached the conclusion that every concord has a ratio which is either multiple or epimoric. the task of the passage that follows, whose results are announced in advance, in [D], is to quantify the ratios of the octave, fifth and fourth. On any plausible dating of the origins of the argument, of course, these ratios were already well known. The author's assumption, however, is that they were known only on the basis of empirical experience

[12] These and related issues are discussed at length in *Harm.* I.1.

[13] See my discussion in 'Reason and perception in Ptolemy's *Harmonics*', in *Harmonia Mundi* ed. R. Wallace and B. MacLachlan (Rome 1991), particularly pp. 116-9.

This content downloaded from 130.43.141.16 on Sun, 12 May 2019 16:06:32 UTC
All use subject to https://about.jstor.org/terms

and practical measurement. His project is the more ambitious one of deriving them by reasoning from first principles, to show that in the light of some necessary or rationally persuasive general truths these *must* be their ratios. The Pythagoreans assign duple ratio, 2:1, to the octave, Ptolemy says in [E], because the octave is the 'finest', καλλίστη, of the concords and the duple is the 'best', ἄριστος, of the ratios. Again, he, or his source, pauses to unravel the notions involved; but this time he makes some attempt to identify the connection between the perceptible feature of the octave that makes it the finest concord and the formal feature of duple ratio that makes it the best ratio. The octave, he says, is finest, 'because it is nearest to equality of pitch'; and the ratio 2:1 is best 'because it alone makes the excess equal to that which is exceeded'. It is hard to make much of this in isolation. A sort of quasi-equality in pitch is being made to correspond to the equality between two elements in the ratio, the smaller term and the difference between the terms; but nothing is said to explain how these two kinds of equality are related. As with the correlation of two kinds of 'simplicity of comparison', Ptolemy repairs the omission elsewhere.[14] But he says nothing to suggest that the explanation is due to Pythagoreans; nor, significantly, is any precedent for it found by Porphyry in his *Commentary*, a work designed in large part to expose Ptolemy's alleged plagiarisms.[15] The likeliest conclusion is the obvious one: that Ptolemy found the present, rather impressionistic correlation in his Pythagorean source, but that the elaborate interpretation of it which he develops elsewhere is his own, and is therefore not relevant to our investigations here.

[14] Ibid., with the continuation to p. 122.

[15] Much of the introduction to Porph. *Comm.* is occupied with Porphyry's debt and that of Ptolemy to their predecessors. After stating and justifying his own policy of borrowing from them freely, he continues: 'In many places I shall not neglect the task of mentioning by name those, whoever they may be, whose demonstrations I shall be using, since I find that the very author I am expounding, while taking much – if not virtually everything – from earlier writers, sometimes indicates the people from whom he took his demonstrations, but elsewhere passes them over in silence.' He goes on to mention, for the second time in the passage, a work by a certain Didymus, who will play a part in our discussion later. (For the identification of this writer as the musical expert said in the *Suda* to have lived in the time of Nero, see *GMW* 2 p. 230. I refer to him below as 'Didymus the younger' to distinguish him from his famous Alexandrian namesake.) 'Indeed, though at many points he paraphrased (μεταγράφων, perhaps 'adapted', or possibly even 'transcribed') the work by Didymus, *On the difference between Pythagorean and Aristoxenian musical science*, he nowhere acknowledged it, and though he took over (or 'modified', μετατιθείς) other things from other people he passed the fact over in silence, as I shall show' (*Comm.* 5.7-15). We need not take Porphyry's πλεῖστα, εἰ καὶ μὴ σχέδον πάντα entirely at face value, but he has a point; and at least his intentions are clear.

This content downloaded from 130.43.141.16 on Sun, 12 May 2019 16:06:32 UTC
All use subject to https://about.jstor.org/terms

Once the ratio of the octave is established the rest is easy. In [F] it is treated as a fact of experience that the note an octave above a given pitch can be reached, starting from that pitch, by ascending through a perfect fourth followed by a perfect fifth (or vice versa, of course), so that the octave is constituted by a combination of the first two concords, as the Greeks conceived such things. The only way of representing the octave ratio, 2:1, as compounded from a succession of two lesser ratios of the appropriate kind is as the product of the first two epimorics ($3:2 \times 4:3 = 2:1$). Hence the perfect fifth, being the larger interval, must be 3:2, and the fourth 4:3. Finally, [G], simple arithmetic will show that the interval of a tone (in alternative modern parlance, the major second), defined as the difference between a fourth and a fifth,[16] has the ratio 9:8; that the octave-plus-fifth is in the ratio 3:1, and that the double octave is in the ratio 4:1. Hence the perceptible concordance of these last two intervals is also reflected in their mathematical form, according to the Pythagorean criteria, since both ratios are multiple. The same cannot be said, as Ptolemy points out in a barbed aside at the end, [H], of the octave-plus-fourth, which sounds like a concord, but has the ratio 8:3, neither multiple nor epimoric. This awkward and notorious fact gives him a useful stick to beat the Pythagoreans with later (13.1-22).

(ii) Argument (b): *Harm.* 12.8-24

[A] They argue to the same conclusion in a more geometrical way, as follows. Let AB, they say, be a fifth, and let BC be another fifth, continuous with the first, so that AC is a double fifth. Since the double

A ____________ 8
B ____________ 12
C ____________ 18

fifth is not concordant, it follows that AC is not multiple, so that neither is AB multiple: but it is concordant, and hence the fifth is epimoric. In the same way they show that the fourth is also an epimoric, a smaller one than the fifth. [B] Again, they say, let AB be an octave and let BC be another octave continuous with the first, so that AC is a double octave. Then since the double octave is concordant, it follows that AC

A ____________ 4
B ____________ 8
C ____________ 16

[16] The definition is accepted by all theorists, whether or not they were prepared to express intervals as numerical ratios: e.g. Aristoxenus *El. Harm.* 21.22, 45.34 – 46.1, 62.1-11, Eucl. *Sect. Can.* prop. 13, Ptol. *Harm.* 16.26-8.

This content downloaded from 130.43.141.16 on Sun, 12 May 2019 16:06:32 UTC
All use subject to https://about.jstor.org/terms

is either epimoric or multiple: but it is not epimoric, since then no mean would fall within it proportionately, and hence AC is multiple, so that AB is also multiple: therefore the octave is multiple. [C] From these things it is plain to them that the octave is duple, and that of the others the fifth is hemiolic and the fourth epitritic, since of the multiples only the duple ratio is constituted by the two greatest epimorics, so that ratios put together from two of the other epimorics are together smaller than the duple, and there is no multiple smaller than the duple.[17]

Ptolemy describes this argument as γραμματικώτερον, 'more geometrical', or 'more diagram-based', though in fact the diagrams are unnecessary. Both the general direction of the reasoning and all its significant details are taken very directly from the treatise called *Sectio Canonis*, attributed by Porphyry to Euclid.[18] That work is a string of interdependent theorems, in which proofs of later propositions depend on earlier ones. Ptolemy's version shows the marks of the difficulties that this strategy creates for an

[17] The text is as follows.

Γραμματικώτερον δὲ προσάγοντες εἰς ταὐτὸν οὑτωσί πως. ἔστω γάρ φασι διὰ πέντε τὸ AB καὶ τούτῳ ἐφεξῆς ἕτερον διὰ πέντε τὸ ΒΓ, ὥστε τὸ ΑΓ εἶναι δὶς διὰ πέντε. καὶ ἐπεὶ ἀσύμφωνον τὸ δὶς διὰ πέντε,

Α	θ′	Α	δ′
Β	ιβ′	Β	η′
Γ	ιη′	Γ	ις′

οὐκ ἄρα διπλάσιον τὸ ΑΓ, ὥστε οὐδὲ τὸ AB πολλαπλάσιον, σύμφωνον δέ, ἐπιμόριον ἄρα τὸ διὰ πέντε. κατὰ τὰ αὐτὰ δὲ καὶ τὸ διὰ τεσσάρων δείκνυσιν ἐπιμόριον ἔλαττον ὂν τοῦ διὰ πέντε. πάλιν ἔστω φασὶ διὰ πασῶν τὸ AB καὶ τούτῳ ἐφεξῆς ἕτερον διὰ πασῶν τὸ ΒΓ, ὥστε τὸ ΑΓ γίνεσθαι δὶς διὰ πασῶν. ἐπεὶ τοίνυν σύμφωνόν ἐστι τὸ δὶς διὰ πασῶν, τὸ ΑΓ ἄρα ἤτοι ἐπιμόριόν ἐστιν ἢ πολλαπλάσιον, ἀλλ' οὐκ ἔστιν ἐπιμόριον – οὐ γὰρ ἄν τις μέσος ἀνάλογον ἐνέπιπτεν – πολλαπλάσιον ἄρα τὸ ΑΓ, ὥστε καὶ τὸ AB πολλαπλάσιον, τὸ ἄρα διὰ πασῶν πολλαπλάσιον. πρόχειρον δὲ αὐτοῖς ἐκ τούτων, ὅτι καὶ τὸ μὲν διὰ πασῶν διπλάσιον, ἐκείνων δὲ τὸ μὲν διὰ πέντε ἡμιόλιον, τὸ δὲ διὰ τεσσάρων ἐπίτριτον. ἐπεὶ μόνος τῶν πολλαπλασίων ὁ διπλάσιος λόγος ὑπὸ δύο ἐπιμορίων σύγκειται τῶν μεγίστων, ὥστε τοὺς ἐξ ἄλλων ἐπιμορίων δύο συντιθεμένους λόγους ἐλάττονας συνίστασθαι τοῦ διπλασίου, μηδενὸς ἐλάττονος ὄντος πολλαπλασίου τοῦ διπλασίου, – – -

But at line 11 the word διπλάσιον, which Düring accepts with one major class of MSS, can hardly be right. The reading πολλαπλάσιον, found in the other MSS, must surely be preferred; and I translate accordingly.

[18] *Comm.* 98.19. The treatise is printed in *MSG*: there are translations in T.J. Mathiesen, 'An annotated translation of Euclid's Division of the Monochord', *J. of Music Theory* 19 (1975) pp. 236-58, and in *GMW* 2. For a very full discussion and edition of the variant texts,with translations, see A. Barbera, *The Euclidean Division of the Canon: Greek and Latin Sources* (Lincoln and London 1991).

This content downloaded from 130.43.141.16 on Sun, 12 May 2019 16:06:32 UTC
All use subject to https://about.jstor.org/terms

excerptor: much of its reasoning hangs on premisses established earlier in the *Sect. Can.*, which are either not stated here or stated but not proved. Hence Ptolemy's presentation is incomplete as it stands, and one could not judge the mathematical credentials of the argument's moves on the basis of his text alone. With one exception, the proofs of the propositions it assumes are in fact all sound: references to the relevant theorems of the *Sect. Can.* are given in the footnotes below. In addition, one logically unacceptable step is made explicitly in the course of the reasoning both here and in the corresponding passage of the *Sect. Can.*; and it is a fatal flaw, for the argument cannot work without it.

The argument begins from an unstated assumption, familiar now from Argument (a) but commended – certainly not proved – on quite different grounds in the introduction to the *Sect. Can.*[19] It is the assumption that the ratios of all concords are either epimoric or multiple. Then, says the argument in [A], since the double fifth (an octave plus one tone) is discordant, it cannot be multiple.[20] This is the explicit mistake of reasoning. The move would be acceptable only on the assumption that all multiple ratios are ratios of concords. But no principle or argument in the *Sect. Can.* underwrites that assumption, and the author of the treatise, if he had been thinking clearly, would certainly have agreed that it is false. The ratio 5:1, for instance, is that of the double octave plus a major third, by Greek standards a discord; and 7:1 is three octaves less a rather large tone.[21] My purpose in referring to the mistake here is not to belabour its perpetrator. Its presence in the text of the *Harmonics* suggests something significant about Ptolemy's attitude to the argument in which it occurs. Ptolemy has a sharp eye for other people's errors of reasoning. A little later he alludes directly, for a different purpose, to the fact that not all multiple ratios are concordant (13.23-14.1). Yet here he allows the move to pass without comment. Taken together with the fact I mentioned earlier, that he presents the argument – again quite uncharacteristically – in so incomplete a form that its credentials cannot be assessed as it stands, this seems a clear indication that Ptolemy is not very interested in Argument (b). Argument (a), on the other hand, interests him a great deal; and with additions and amendments it becomes the basis for his own approach to the derivation of the ratios (I.7, especially

[19] 149.17-24 (*MSG*).
[20] *Sect.Can.* prop. 11.
[21] On this issue see my 'Methods and aims in the Euclidean Sectio Canonis', *JHS* 101 (1981) pp.4-5. Attempts have been made to find a legitimate interpretation for the argument of prop. 11: see especially Bowen's paper cited in n. 7 above. I am not persuaded.

This content downloaded from 130.43.141.16 on Sun, 12 May 2019 16:06:32 UTC
All use subject to https://about.jstor.org/terms

15.18-16.28). To Ptolemy, at least, they appear to be arguments of very different sorts.

We shall return to this point. Meanwhile, suppose we too let the mistake pass. Then *if* the double fifth is not of multiple ratio, the argument continues, neither is the fifth itself. The inference is correct.[22] By similar reasoning, the ratio of the fourth is not multiple either. But both are concords: hence, by the original principle, both are epimoric.[23]

The double octave, by contrast, is a concord. This fact provides the basis for section [B]. It follows that it is multiple or epimoric. But it cannot be epimoric, since it contains a mean proportional, at the octave, and there is no mean proportional between terms in an epimoric ratio. That is, if A:B is epimoric, there is no ratio of integers x:y such that $(x{:}y) \times (x{:}y) = A{:}B$. Consequently no interval whose ratio is epimoric can be divided into two or more equal intervals each of which corresponds to a ratio of integers. Hence the double octave must be multiple, since it is a concord and *can* be so divided.[24] This reasoning, again, is sound. If the double octave is multiple, Ptolemy continues, then so is the octave itself: the reasoning is suppressed, but this inference too is correct.[25]

Now everything is plain sailing. The reasoning in [C] will be roughly parallel to the equivalent part of Argument (a), though couched in Euclidean terms and starting from different premisses. Summarily, since the octave is multiple and is composed of the fifth and the fourth, which are epimoric, their ratios must be 2:1, 3:2 and 4:3 respectively. No other ratios will fit the bill.[26]

(iii) Argument (c): *Harm.* 12.24-27

> Since the tone is accordingly shown to be epogdoic (9:8), they reveal that the half-tone is unmelodic, because no epimoric ratio divides another proportionately as a mean, and melodic magnitudes must be in epimoric ratios.[27]

[22] *Sect. Can.* prop. 11, grounded in a theorem proved at prop. 5.

[23] *Sect. Can.* prop. 11.

[24] *Sect. Can.* prop. 10. The crucial theorem about mean proportionals is proved at prop. 3.

[25] *Sect. Can.* prop. 10, based on a theorem proved at prop. 2.

[26] *Sect. Can.* prop. 12.

[27] The text is as follows.

καὶ τοῦ τόνου δὲ ἀκολού-
θως ἐπογδόου δειχθέντος, ἀποφαίνουσι τὸ ἡμιτόνιον ἐκμελές, ἐπεὶ μηδ'
ἄλλος τις πάλιν ἐπιμόριος μέσος ἀνάλογον διαιρεῖται, δέον δὲ ἐν λόγοις
27 ἐπιμορίοις εἶναι τὰ ἐμμελῆ.

This content downloaded from 130.43.141.16 on Sun, 12 May 2019 16:06:32 UTC
All use subject to https://about.jstor.org/terms

This little argument needs some attention before Argument (b) can be properly reviewed. I referred to it earlier as an appendix. Ptolemy makes it continuous with Argument (b): I shall explain shortly why I think it right to deal with it separately. It has some surprising features. In the immediately preceding passage Ptolemy has faithfully reproduced the reasoning of the *Sect. Can.*, though in an abbreviated form. The present argument has affinities with that treatise's Proposition 16. But if it is drawn from that source it is a distressingly garbled version; and it introduces both a conception and a principle which are not in the *Sect. Can.* at all. The Euclidean argument is not designed to show that no epimoric ratio can be compounded from equal epimoric ratios. It proves the more general proposition that no epimoric ratio can be compounded from equal ratios of integers of any sort.[28] Ptolemy has in fact just used this point in part [B] of Argument (b). From the Euclidean point of view, the half-tone is not unmelodic but non-existent: 'the tone', asserts Prop. 16, 'will not be divided into two or more equal intervals'. And though of course if the half-tone is not a ratio of integers, then *a fortiori* it is not an epimoric ratio, nevertheless the way the point is put in Argument (c) is seriously misleading.

Further, the *Sect. Can.* is quite innocent of the notion of 'melodic' (ἐμμελῆ) intervals or magnitudes, on which Ptolemy's form of the argument depends. It uses neither the term ἐμμελής itself, nor any other word in a comparable sense. In Ptolemy's usage, 'melodic' intervals are those intervals, smaller than the fourth, which form the elementary 'scalar' steps between adjacent notes in well-formed attunement. Though the *Sect. Can.* says nothing about these intervals as a class, several members of it are mentioned and (by clear implication) quantified in its theorems, and two important instances have ratios which are not epimoric:[29] yet Argument (c) asserts, as a fundamental principle, that all melodic intervals must have such ratios. The rule enunciated in Argument (c) is thus quite radically un-Euclidean; and we have excellent reasons for refusing to believe that it comes from the same stable as Argument (b). I shall suggest that it is much more likely to be associated with the ideas of Argument (a). But to show that, we need to take stock a little.

[28] The mathematical proposition is proved at prop. 3, and applied to the issue of the division of the tone at prop. 16. It is put to work again in prop. 18, in connection with a different interval.

[29] These are the ditone involved in the discussion of enharmonic divisions in prop's 17-18, and the small interval left within a perfect fourth after two steps of a whole tone each (often called the λεῖμμα), which emerges from the construction of a diatonic system of attunement in prop's 19-20. Their ratios are 81:64 and 256:243 respectively.

This content downloaded from 130.43.141.16 on Sun, 12 May 2019 16:06:32 UTC
All use subject to https://about.jstor.org/terms

Argument (a) depended crucially on the attribution to both concords and ratios of value-loaded properties. Concords are 'finer' intervals, and epimorics and multiples are 'better' ratios. More specifically, the 'finest' interval is the octave, and the 'best' ratio is duple ratio, 2:1. If we allow these claims, and allow also the correlation made between these positively valued characteristics of intervals on the one hand and of ratios on the other, then the way in which Argument (a) derives the specific ratios of the concords has the merit that it works.

By contrast, Argument (b) uses no value-loaded terms. The reasoning by which the author of the *Sect. Can.* seeks to establish the link between concords and multiple or epimoric ratios, in his introduction, is of quite another sort: it is value-neutral, and on what seems to me the most natural interpretation it is also decidedly unconvincing.[30] From there on, the *Sect. Can.* and Argument (b) try to derive the sizes of the concords from theorems belonging wholly within the mathematics of ratio, as we would understand that conception; and unsurprisingly they fail. The task is in fact impossible. It only seems to have been done because of a perhaps inadvertent conjuring-trick, involving the slip in the first step of the reasoning expounded by Ptolemy, and in Prop. 11 of the *Sect. Can.*

We have already seen some reasons for wanting to detach Argument (c) from Argument (b). At best, one strand in it is a confused echo of something in the *Sect. Can.* It employs a conception of which there is no trace in that work, and it asserts a rule which the *Sect. Can.* neither mentions nor follows. The restriction of melodic intervals to those of epimoric ratio, under that rule, would fit well, however, within the evaluative scheme of Argument (a). Epimoric ratios are 'better' than epimerics; and the theorist of Argument (a) will need some way of marking an evaluative distinction between 'melodic' intervals and 'unmelodic' ones, those that can have no place in melody. (The Euclidean treatise makes no attempt to draw a distinction of this sort.) It will obviously make sense to identify the melodic intervals with the lesser epimorics, those smaller than 4:3 (which is the strategy followed by Ptolemy himself later on), and to consign unmelodic ones to the scrap-heap of the epimerics. It is easy to guess why Ptolemy nevertheless chose to tack Argument (c) on to Argument (b). It is because Argument (c) alludes, however confusedly, to the theorem about mean proportionals which played a part in Argument (b), but which Argument (a) did not mention. But then if (c) really belongs with (a) and not with (b), I need to give grounds for believing that the theorem was in the repertoire of the

[30] See my paper, n. 19 above, pp. 2-3, and *GMW* 2 p. 193 n. 8: for a rival interpretation see Bowen (n. 7 above) pp. 176-182.

This content downloaded from 130.43.141.16 on Sun, 12 May 2019 16:06:32 UTC
All use subject to https://about.jstor.org/terms

exponents of Argument (a), though it is not used or referred to there. We shall come back to that issue shortly.

SECTION 2 The origins of Arguments (a) and (c)

Arguments (a) and (b), I have suggested, have quite different assumptions and approaches, and different presuppositions about the kinds of concept on which mathematics and scientific harmonics can draw. As a consequence, there can be little doubt that they come from different sources. I have also made a case for the view that Argument (c) cannot belong with (b); and I have sketched a reason for thinking that it might have originated in the same context as (a). We know exactly where (b) comes from, as did Porphyry, who quotes almost the whole of the *Sect. Can.* in his commentary on this passage,[31] though it would be rash to assume that he is right in attributing it to Euclid: the treatise's date and authorship are disputed.[32] But these issues need not concern us here. Let us consider instead whether anything can be established about the pedigree of Argument (a), and of its possible appendage, Argument (c).

It will be convenient to begin with (c), and its thesis that all melodic intervals must be epimoric. We can identify by name three theorists earlier than Ptolemy who were known by him to have subscribed to this rule or a slightly modified version of it. In reverse chronological order they are Didymus the younger in the first century AD, Eratosthenes in the third century BC, and Archytas in the first half of the fourth.[33] In principle, then, any of these might have been the author of Argument (c).

On general grounds, however, neither Didymus nor Eratosthenes is a likely candidate. For one thing, nothing we know about them encourages the belief that either made substantial original contributions to harmonic theory, though they have other virtues. Again, neither is a very plausible

[31] Porph. *Comm.* 98.14-103.25.

[32] For references to older debates on the treatise's authorship see Mathiesen (n. 16 above) p. 253 n. 1. More recent discussions include A. Barbera, 'Placing *Sectio Canonis* in historical and philosophical contexts', *JHS* 104 (1984) pp. 157-61, and his full-scale presentation of the evidence in the work cited in n. 18 above. I am not yet convinced by his arguments for the view that the treatise was put together gradually, at various dates, and specifically not by his attempts to detach propositions 17-20 from the preceding theorems.

[33] For the harmonic divisions proposed by these theorists see *Harm.* II.14. There are deviations from the rule in the divisions of both Eratosthenes and Archytas. As will become clear immediately, Eratosthenes' approach need not be pursued here (see *GMW* 2 p. 346 n. 117, p. 349 n. 125). On the anomalous features of Archytas' divisions see the discussions mentioned in n. 35 below. On the identity of this Didymus see n. 14 above.

This content downloaded from 130.43.141.16 on Sun, 12 May 2019 16:06:32 UTC
All use subject to https://about.jstor.org/terms

candidate for the label 'Pythagorean', though the term is admittedly used as loosely and haphazardly in the context of harmonics as it often is elsewhere. Ptolemy says a good deal about Didymus, without suggesting any connection with what he understood as Pythagoreanism, nor does he attach the title when he mentions Eratosthenes. In Eratosthenes, in fact, he shows little interest; and we have seen that he had every reason to be interested in the author of Argument (c), since its rule is one that he adopts, with modifications, as the central principle of his own approach to harmonic division (see especially I.15). On the other hand he describes Archytas not only as a Pythagorean, but as 'of all the Pythagoreans the most dedicated to the study of music' (30.9-10), and he analyses aspects of his work in detail. Didymus, however, may enter the picture in a different role. Porphyry draws on him extensively as a historian and classifier of schools of harmonic theorists, and implies that Ptolemy did so too; and one of his remarks leaves little room for doubt that it was in Didymus' writings that both Porphyry and Ptolemy found their main information about Archytas.[34]

There are good reasons, too, for thinking that whoever first enunciated the rule stated in Argument (c) was someone who was prepared, like the author of the argument itself, to support his views with reasoning grounded in theory. Though the rule is merely stated in Argument (c) as it stands here, not argued for, my point is that in the absence of substantial argumentative support the principle would be hopelessly implausible. On the face of it it is merely false. The simplest practical way of tuning an instrument to an eight-note scale in which octaves, fifths and fourths are fundamental is one that inevitably involves breaches of the rule; and there is good evidence that the Greeks knew this method and regularly used it.[35] Further, there were influential theorists for whom such an attunement, one in breach of the rule in Argument (c), was not only acceptable but theoretically perfect: they include Philolaus, Plato, the author of the *Sect. Can.*, and any number of later writers of a more or less Platonist persuasion.[36] Anyone who insisted on the rule in Argument (c) would therefore need powerful reasons to support his position, and to give him grounds for believing it in the first place himself.

[34] See n. 14 above, and Porph. *Comm.* 3.13-14, 5.11-14, 25. 4-6, 26. 6-28.26. The remark I refer to is at *Comm.* 107.15. It is misconstrued and mispunctuated by Düring (the comma should come after ἱστοροῦσιν, not before). See *GMW* 2 p. 34 n. 25.
[35] See for instance Aristoxenus *El. Harm.* 55.3-12, Eucl. *Sect. Can.* prop. 17, Ptol. *Harm.* 40.8-17.
[36] Philolaus DK44 B6, Plato *Tim.* 35b-36b, Eucl. *Sect. Can.* prop's 19-20; for later examples see e.g. Theon Smyrn. 90.22-92.25, Arist. Quint. *De Mus.* 96.18-28.

This content downloaded from 130.43.141.16 on Sun, 12 May 2019 16:06:32 UTC
All use subject to https://about.jstor.org/terms

The earliest proponent of the rule cannot then have founded it on observation or tradition, with which it was inconsistent. Its basis must have been theoretical and argumentative from the start. Its earliest known proponent is Archytas, a writer with an undeniable bent towards mathematical theory and abstract argumentation. Despite the large gaps in our evidence, it is not at all likely that anyone before him subscribed to the rule. There are no signs of speculations that might have a bearing on the matter in the fifth century, except in the work of Philolaus; and his views are flatly at odds with the one stated here.[37] The chances are, then, that Archytas not only subscribed to the rule but originated it, and that he also offered theoretical arguments to support it.

I am obviously insinuating that if we are looking for a source for Argument (c), the finger points in Archytas' direction. Equally obviously I have not proved it. But as soon as we start to think about Archytas as a possible source, other pieces of the puzzle begin to fall into place. He certainly adopted a form of the rule that elementary relations in an attunement should be epimoric, even though Ptolemy accuses him of applying it inconsistently.[38] Ptolemy also attributes to him a line of thought that can be construed as an attempt to justify the rule: we shall return to this shortly. Next, the originator of Argument (c) called on a version of the theorem showing that there is no mean proportional between terms in an epimoric ratio. This seemed to pose a problem for my suggestion that Argument (c) does not go with (b) but does go with (a), since (b) uses that theorem and (a) does not allude to it. But we are told by a different source, Boethius, late and not wholly reliable though he may be, that Archytas himself propounded a theorem to this effect;[39] and it is worth noting that the Archytan version of the theorem, as given in Boethius, is slightly different from that of the *Sect. Can.*[40] It seems a reasonable guess, then, that this theorem was transmitted in two distinct traditions, one through the *Sect. Can.*, the other, I suggest, through Didymus, who is probably Ptolemy's and Porphyry's source for Archytan material, and might well be Boethius' too, at one or two re-

[37] The principal text is again DK44 B6. For a full study of Philolaus see now C.A. Huffman, *Philolaus of Croton: Pythagorean and Presocratic* (Cambridge 1993).
[38] *Harm.* 32.1-3, cf. 30.13-31.18. For attempts to interpret Archytas' constructions in a manner consistent with a version of the principle see my 'Archita di Taranto e l'armonia pitagorica' in A.C. Cassio and D. Musti (eds), *Tra Sicilia e Magna Grecia* (Naples 1989) pp. 159-78, and *GMW* 2 pp. 46-52.
[39] *Inst. Mus.* III.11.
[40] See W.R. Knorr, *The Evolution of the Euclidean Elements* (Dordrecht 1975), ch. 7.

This content downloaded from 130.43.141.16 on Sun, 12 May 2019 16:06:32 UTC
All use subject to https://about.jstor.org/terms

moves.[41] In that case if Archytas is indeed the inventor of Argument (c), there is no need for it to be dependent on the *Sect. Can.* and on Argument (b). In view of what I said earlier, this is a conclusion to be welcomed.

But if Archytas is behind Argument (c), we can find another reason, stronger than the rather impressionistic one I used earlier, not only for detaching it from Argument (b), but for associating it closely with Argument (a). Later in the *Harmonics* Ptolemy gives a brief account of the way in which Archytas justified his restriction of melodic intervals to those of epimoric ratio. It was through the thesis that 'the commensurability of their excesses is a characteristic of the nature of melodic intervals' (30.12-13). The sense of this opaque remark is that the difference between the terms of the ratio (the 'excess' of one over the other) in any melodic interval must be a 'measure', a simple part or factor, of each of the terms; and from this it will follow that the ratio must be epimoric.[42] Whatever the speculations lurking behind this contention may have been, it draws on precisely the feature of epimoric ratios that was highlighted in Argument (a) – that the difference between the terms is a simple part of each term, and so provides a 'measure' for them. This is what gives the comparison of such terms the 'simplicity' from which the ratio derives its excellence. The rule in Argument (c), then, was defended by Archytas on grounds that figure prominently in Argument (a). The rule and its justification almost certainly originated with him, and it seems very probable that Argument (a) did too. There is no trace of that argument, or of this justification for the rule, in any extant source between Archytas and Ptolemy. I submit that we have good reasons for assimilating Argument (c) to the ambience of Argument (a), and for attributing both to Archytas.

There are two other suggestive pointers to this conclusion. I have already remarked that Archytas is much the most likely candidate among theorists drawn on by Ptolemy for the title 'Pythagorean'. It is also the case that the

[41] Boethius' source is generally agreed to be Nicomachus: see the introduction by C.M.Bower to his translation of Boethius, *Fundamentals of Music* (New Haven and London 1989), especially pp. xxvi-xxvii. If Nicomachus drew on Didymus it seems likely that he did so directly, without depending on intermediary reports. Didymus' treatise was evidently still available in Ptolemy's time, and indeed in Porphyry's.

[42] This consequence follows even from the stipulation that the difference between terms is an integral factor of just one of the terms (e.g. *Harm.* 16. 18-19). It can be shown straightforwardly both that it must be a factor of the other term too, and that the ratio is epimoric. Let the ratio be A:B, where A is the larger term. Let the difference (A-B) be an integral factor of B. Then B = m(A-B), where m is an integer. Hence A = m(A-B) + (A-B) = (m+1)(A-B), and since (m+1) must also be an integer, (A-B) is an integral factor of A. Hence A:B = (m+1)(A-B):m(A:B) = (m+1):m, and A:B is therefore an epimoric ratio.

This content downloaded from 130.43.141.16 on Sun, 12 May 2019 16:06:32 UTC
All use subject to https://about.jstor.org/terms

general tone of Ptolemy's presentation of Argument (a) seems to fit his attitude to Archytas better than it fits his attitude to anyone else. When he mentions his predecessors it is almost always to criticize them. But though he rejects some aspects of Argument (a), he is uncharacteristically flattering about others. The first principle of its proponents' method is 'entirely appropriate', he says (11.8); and another step in their procedure is 'very rational' (11.20-21). The only comparable accolade elsewhere in the *Harmonics* is given to Archytas. 'Archytas of Tarentum, of all the Pythagoreans the most devoted to the study of music, tried to preserve what follows the principles of reason not only in the concords but also in the divisions of the tetrachords . . .' (30.9-12). The echoes of this remark (which amounts in Ptolemaic terms to a resounding fanfare) are admittedly damped down almost at once, as Ptolemy proceeds to criticism; but even there he allows that Archytas 'is in most cases well in control of this sort of thing' (30.14-15), only occasionally lapsing into error. And it is a fact that the approach Ptolemy attributes to Archytas is recognisably closer to his own than is that of any other individual he mentions.

Secondly, the statement I quoted above (30.9-12) implies that Archytas 'tried to preserve what follows the principles of reason' in connection with the concords, among other things. In the context this can only mean that Ptolemy believed him to have postulated certain rational principles to account for the concords' possession of the ratios they have; and unless he is suppressing some quite different suggestion he thought Archytas had made, the arguments he used can only be those set out in *Harm.* I.5. They cannot be those of Argument (b) and the *Sect. Can.*, not only because Porphyry unambiguously attributes that treatise to Euclid (he might be wrong, or the Euclidean author might in turn be dependent on Archytas, as has sometimes been thought), but because at several points that work is flatly inconsistent with what Ptolemy and Porphyry, at least, tell us of Archytas' views.[43] This

[43] The enharmonic and diatonic divisions of the *Sect. Can.* (Prop's 17-20) are quite different from those attributed by Ptolemy to Archytas (*Harm.* I.13, II.15), and inconsistent, as we have seen, with the restriction of melodic intervals to those with epimoric ratios. The two authors' theories about the physical basis of pitch are also clearly distinct: compare the theory of Archytas fr. 1 (DK47 B1 = Porph. *Comm.* 56.5-57.27) with that of *Sect. Can.* 148.3-149.8 (*MSG*). Hence anyone who wishes to associate the *Sect. Can.* closely with Archytas must deny the evidence of both Ptolemy and Porphyry (I am grateful to the editor of *Phronesis* for helping me to see the point in this light). But the authenticity of the divisions recorded by Ptolemy has never to my knowledge been questioned; and though fr. 1 has been much debated there is currently, I think, a consensus in its favour. See especially A.C. Bowen, 'The foundations of early Pythagorean harmonic science: Archytas, fragment 1', *Ancient Philosophy* 2 (1982), 79-104, and C.A. Huffman, 'The authenticity of Archytas fr. 1', *CQ* 35 (1985), 344-8.

This content downloaded from 130.43.141.16 on Sun, 12 May 2019 16:06:32 UTC
All use subject to https://about.jstor.org/terms

leaves Argument (a) as the only candidate; and we are forced to conclude that in Ptolemy's opinion its author was Archytas.

This completes my attempt to locate the source of Arguments (a) and (c). If I am right we have some new morsels of Archytan material to add to our modest corpus. It is, at least, material that was taken to be Archytan by Ptolemy, and presumably by his source. We have no direct way of deciding whether their belief was true. The arguments are obviously not presented in Archytas' own words – the dialect alone shows that.[44] But their sense fits comfortably enough with what else we know of his work. To judge by the relation of Argument (b) to the *Sect. Can.*, and of other Ptolemaic paraphrases to their sources, where these are known, Ptolemy was probably following his immediate authority rather closely. I have already said that I take this authority to be the younger Didymus; and our information about this musical Didymus suggests both that he was competent and reliable in his accounts of earlier writings in harmonic theory, and that he had access to some rather unusual documents.[45] The only reason for doubting the truth of Ptolemy's implied attribution lies in the general attitude of scepticism with which modern scholars (often with ample justification) are inclined to approach Hellenistic or later reports about 'Pythagorean' doctrines. I have no wish to dissipate or ignore this pervasive cloud of suspicion. I merely point out that I can find no solid grounds for mistrust in the present case, and some tolerably persuasive reasons for belief.

Now Archytan material of this kind, if such it be, should at least interest specialists in ancient harmonics; but we are rare birds. In my short final section I shall try to suggest, though very sketchily, a way in which it might be significant also for mainstream students of ancient philosophy.

SECTION 3 Archytan mathematics and the philosophy of Plato

I am not concerned here with questions about episodes in the traditional biographies of Plato which bring him into personal contact with Archytas.

[44] Unlike his commentator Porphyry, Ptolemy rarely quotes his sources verbatim. In his discussion of Archytas at I.13-14, for instance, he reports and paraphrases Archytas' views exclusively in his own (sometimes idiosyncratic) terminology. Where he does quote his predecessors' terms directly it is usually with critical – even satirical – intent, as for instance with the Pythagorean uses of the words ὁμοιότης and ἀνόμοιος at 14.4ff. (See Porphyry's report at *Comm.* 107.15-108.21 for a fuller account of the procedures derided here by Ptolemy.)

[45] See n. 31 above. Whether or not Ptolemy's borrowings from Didymus were extensive, as Porphyry suggests, he was sufficiently impressed by him to devote careful discussion to some of his original ideas (*Harm.* II.13), as well as recording his harmonic divisions (II.15).

This content downloaded from 130.43.141.16 on Sun, 12 May 2019 16:06:32 UTC
All use subject to https://about.jstor.org/terms

In particular, I shall make no judgement about the reliability of the famous Seventh Letter. My intention is only to gesture in a broad, schematic way at certain affinities between these philosophers' ideas and objectives in the mathematical domain, and to suggest ways in which Ptolemy's Archytan material can be related to them. The evidence comes from relatively uncontroversial sources – from Plato's dialogues and from a few Archytan fragments and testimonia that are standardly accepted as authentic.

In *Republic* VII Plato's Socrates castigates the Pythagoreans, among others, for neglecting the most important tasks to which harmonic theorists should set their hands, or rather, their minds (530c-531c). His criticisms pose some interpretative problems, but for present purposes we need only the gist of his two main contentions. The first is that they pay too much attention to what they hear: they seek to quantify concords as they present themselves to the hearing in musical practice, rather than the relations of mathematical concordance between numbers, which are accessible only to the mind. Secondly and relatedly, they do not 'ascend to προβλήματα', to discover not only 'which numbers are concordant and which are not', but also 'in each case, why' (531c, cf. 530e-531a). These are matters I have discussed elsewhere, [46] and the remarks that follow will be shamelessly dogmatic on several issues which I do not wish to reopen now. The central point is this. It is very likely indeed that Socrates' comments on Pythagorean theorists were perfectly apt at the period in which the dialogue is dramatically set. But considered in relation to the time at which it was written, and hence in some possible connection with the work of Archytas, at least part of the criticism misses its mark. Though Archytas was indeed concerned to quantify systems of attunement in actual use, ones that are heard, [47] we know also that he offered a principle, in the form of a theory about means and proportions, which was evidently designed for precisely the task that Plato's Socrates has proposed – the task of explaining why, from a mathematical point of view, certain groups of relations are well formed, mathematically coordinated or 'concordant', while others are not.[48] Not only that: we know also that Plato approved of this principle, if not of Archytas' way of applying it, since he uses it himself as the foundation for

[46] '*Symphonoi arithmoi* : a note on *Republic* 531c1-4', *Classical Philology* 73 (1978) pp. 337-42.

[47] This is clear from *Harm.* I.13. See also R.P. Winnington-Ingram, 'Aristoxenus and the intervals of Greek music', *CQ* 26 (1932) pp. 195-208.

[48] DK47 B2 = Porph. *Comm.* 93.6-17. For discussions of the ways in which he applied the theory to the subject of harmonic division, and comparisons with Plato's procedure, see the papers cited in n. 38 above.

This content downloaded from 130.43.141.16 on Sun, 12 May 2019 16:06:32 UTC
All use subject to https://about.jstor.org/terms

his division of the World Soul in the *Timaeus*.[49] We might even guess that the remarks in the *Republic* are designed partly as a compliment to Archytas rather than a criticism, though for reasons of verisimilitude, given the dialogue's dramatic date, this could not be made explicit. It would be an indication that the science which Socrates advertises as a most impressive and very difficult enterprise is after all not impossible, since a slightly misdirected version of the science he imagines was in the field already by the time the *Republic* appeared. Close associates of Plato would not, perhaps, have had much trouble in picking up the reference to it.

Now if I am right in attributing our Argument (a) to Archytas, it suggests another reason why Plato might have been in sympathy with his mathematical ideas; and it could tell us something significant about mathematics as Plato and Archytas conceived the science – and as the Euclidean source of Argument (b) did not. Many passages of Plato, especially in *Republic* and *Timaeus*, show that there was a close connection in his mind between mathematical truths and truths about excellence and value. The precise nature of the connection is admittedly elusive. There are notorious difficulties, for instance, in interpreting the relation between the ὑποθέσεις of mathematicians and the ἀνυπόθετος ἀρχή that is the Good itself, in the simile of the Divided Line (*Rep*. 510b4-511d5), or in understanding just how the mathematical disciplines outlined in *Republic* VII constitute stepping-stones to the dialectic through which philosophers may grasp the nature of the Good and the Beautiful. If our mathematics is of the hard-nosed Euclidean type, in which mathematical and evaluative concepts belong to sharply distinct domains, these questions are bound to be troublesome, if not insoluble.

But there is no need to saddle Plato with this conception of mathematics – not, at least, of mathematical studies pursued in the manner he thought best. The arguments we have been considering give some indirect encouragement to the view that his stance was radically different. For if Plato approved of Archytan mathematics, and if Argument (a) is a specimen of Archytas' mathematical reasoning, then the mathematics of which Plato approved had evaluative conceptions built into it, and there is no categorical gap between its sphere of operations and that of the Platonic dialectician.

Let us recall that Plato's Socrates speaks directly of certain classes of numerical relation as σύμφωνοι, 'concordant', as though this term, borrowed though it plainly is from the acoustic domain, could appropriately be used in mathematics itself. The 'concordance' of certain ratios is to be established and explained through strictly mathematical reasoning, conducted by the mind alone without reference to the sphere of perceptible sound

[49] *Tim*. 36a: cf. e.g. Nicomachus *Ench*. 8, Arist. Quint. *De Mus*. III.5.

This content downloaded from 130.43.141.16 on Sun, 12 May 2019 16:06:32 UTC
All use subject to https://about.jstor.org/terms

(*Rep.* 531c). The word σύμφωνος undoubtedly carries evaluative loading, as passages in the *Republic* itself make plain.[50] In that case it is implied that some relations between numbers, simply as such, are intrinsically better than others, just as is presupposed in Argument (a). That is a *mathematical* description of them; and the task of the harmonic theorist is to discover which these 'better' relations are and what makes them so. The arguments I am now attributing to Archytas would confirm that this way of talking is not just a fanciful interpolation from outside mathematics, but reflects conceptions and operations that are involved in the procedures and the reasoning of the science itself. Argument (a) is something like a theorem of this mathematics; and it makes essential use of evaluative premisses concerning mathematical entities, ratios themselves, quite apart from their instantiation in audible sounds. There are better and worse ratios, and one ratio is the best of all, and serious conclusions follow from these propositions. They are not mere pieces of external decoration.

The suggestion that Plato's style of mathematics involves evaluative conceptions has of course been made before.[51] So far as I know, however, scholars have not tracked down actual specimens of detailed mathematical argumentation, actual theorems current among Plato's associates, which draw explicitly as this one does on evaluative premisses as parts of their working. For what they are worth, I offer them Argument (a) and its adjunct, Argument (c); and I must leave to people better versed than I am in the mysteries of Academic mathematics to pursue the suggestion further.[52]

Department of Classics, University of Otago

[50] Clear examples are 401d2, 430e3, 441e9, 442c10, 591d2, d7.

[51] See for instance J.C.B. Gosling, *Plato* (London and Boston 1973) pp. 100-107; A.P.D. Mourelatos, 'Plato's Real Astronomy: *Republic* 527d-531d', in J.P. Anton (ed.) *Science and the Sciences in Plato* (New York 1980) pp. 33-73; and his 'Plato's Science – his view and ours of his' in the collection edited by A.C. Bowen (n. 7 above), pp. 11-30. Essential reading on this topic (though its main focus is elsewhere) is M.F. Burnyeat, 'Platonism and Mathematics: a Prelude to Discussion', in A. Graeser (ed.), *Mathematics and Metaphysics in Aristotle* (Bern 1987), pp.213-240. Burnyeat operates at a level far subtler than anything I attempt in this closing section. But I construe his pp.238-40, in particular, as indicating that on what I treat here as the crucial issue, Burnyeat's rapier and Barker's bludgeon are ranged on the same side.

[52] An earlier version of this paper was delivered at a conference at the University of Melbourne in 1991, and I have benefited from the comments of Paul Thom, Kim Lycos, Dougal Blyth and others on that occasion. My thanks to them; to the University of Warwick for the period of leave that allowed me to devote some months of that year to Ptolemy; to the University of Queensland for the funds and to their Department of Classics for the hospitality which allowed me to pursue my quarry in such idyllic surroundings.

This content downloaded from 130.43.141.16 on Sun, 12 May 2019 16:06:32 UTC
All use subject to https://about.jstor.org/terms